J. Fluid Mech. (1970), *vol.* 42, *part* 2, *pp.* 337–347
Printed in Great Britain

Inviscid film flow over an inclined surface originated by strong fluid-injection

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(Received 1 August 1969)

This paper considers inviscid film flows that originate by injection of fluid through the bounding surface. In the first part of the paper plates of an average inclination are considered. Power-law injection rates are studied in some detail. Later it is shown how the analysis has to be modified for almost horizontal plates.

1. Introduction

If a liquid is being blown at a continuous rate through an inclined porous surface it will form a film that flows in a downward direction under the action of the longitudinal component of the force of gravity. If the rate of mass transfer is large and the fluid has a small viscosity, it may be expected that the film will be mainly inviscid. Several recent papers have dealt with the problem of introducing fluid into an outer stream by means of strong blowing. Acrivos (1962), Watson (1966), Cole & Aroesty (1968) and Elliot (1968) concluded that there exists an inviscid boundary layer near the surface. At some distance from the surface there is a viscous shear layer beyond which the fluid is again flowing inviscidly. A singular perturbation technique was employed by the above authors to match the viscous and inviscid flows.

In the present paper we study strong blowing resulting in a film flow. However, we will not be interested in the first place in the influence of viscosity on a mainly inviscid flow. The problems solved in this paper concern purely inviscid flows only. Despite the inviscid character of the flow we will apply the boundary-layer concept to describe the film. This means that the longitudinal component of the momentum equation is of primary importance. The component in the normal direction only yields a simple relation for the pressure. For plates with an average inclination ($\alpha \sim \frac{1}{4}\pi$) the zeroth perturbation of the pressure may be taken equal to zero, which is similar to traditional boundary-layer theory with zero longitudinal pressure gradient (Blasius's flow). Perturbations about this zeroth order are proportional to the Froude number, which actually is the ratio of the injection velocity and the velocity of free fall based upon the longitudinal component of the force of gravity. These Froude number perturbations are complicated by the fact that generally the zeroth-order solutions are singular at the outer edge, i.e. at the zeroth-order position of the outer edge. The true position of the outer edge will be farther away from the surface, since the second

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momentum equation represents an adverse pressure gradient. Therefore, the singularity would be inside the field of interest if one would expand regularly. By introducing strained co-ordinates, the singularity may be placed outside the flow field. Thus we will use Lighthill's technique to develop Froude number expansions.

If the plate is almost horizontal the pressure is no longer zero to zeroth order. The adverse pressure gradient becomes so important that it can no longer be regarded as a small perturbation. This case is considered in detail in the last part of the paper. Uniform blowing will receive special attention. If the result is expanded for average inclination, complete agreement is found with the perturbation solution of the previous section.

2. Average inclinations

Let us consider a semi-infinite flat plate, the leading edge ($x = 0$, $y = 0$) being its highest point. The co-ordinate x measures distance along the plate, y measures distance normal to the plate, u and v are the velocity components in the x and y directions respectively. Let g denote the component in the x direction of the acceleration due to gravity and $-g\cot\alpha$ that in the y direction. Then, if $\tilde{p}$ is the pressure and ρ the density, the governing inviscid equations for an incompressible fluid are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{\tilde{p}}{\rho}\right)+g, \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{\tilde{p}}{\rho}\right)-g\cot\alpha. \tag{3}$$

Boundary conditions are imposed at the wall,

$$v = v_w(x), \quad u = 0 \quad \text{at} \quad y = 0, \tag{4}$$

which denote that the fluid is injected normally at a certain rate. At the outer edge of the film, $y = y^e(x)$, which is defined by $\psi(x, y^e) = 0$, we have

$$\tilde{p}(x, y^e) = \tilde{p}^e = \text{constant}.$$

An estimate of the thickness of the film can be obtained by equating the amount of fluid that flows downwards through the film at x per unit of time and the amount of fluid that is being blown through the plate between 0 and x during the same time. The order of magnitude of the former is $y^e(x)\,(gx)^{\frac{1}{2}}$ since the inviscid film can fall freely in a downward direction. As the latter amount is of the order $v_w(x)\,x$ we have

$$y^e(x)/x \sim v_w(x)/(gx)^{\frac{1}{2}}, \tag{5}$$

which one could consider to be a Froude number. If this is much less than unity, the film is thin and thus boundary-layer approximations may be applied to (1)–(3). Without going into much detail here, it may suffice to mention, referring to the original work of Prandtl, that to first order in the boundary-layer approximation, (3) reduces to $\partial\tilde{p}/\partial y = 0$: the pressure is approximately constant through the boundary layer. Here this will be valid if $\cot\alpha$ is not large. By expanding about the zeroth order one may improve the boundary-layer solution.

In the present chapter this problem will be considered for power-law injection rates, i.e. $v_w(x) = Nx^k$. For these injection distributions the zeroth-order equations admit a similarity solution. Since it can be expected that the boundary-layer solution becomes more accurate as the relative thickness of the film (5) becomes smaller, it is reasonable to introduce this relative thickness as a small perturbation parameter. After introducing the stream function ψ in the usual way, we perform the following transformation:

$$\psi = Nx^{k+1}f(\eta,\xi), \quad \frac{\tilde{p}-\tilde{p}^e}{\rho} = gxp(\eta,\xi), \quad \xi = \frac{Nx^k}{(gx)^{\frac{1}{2}}}, \quad \eta = \frac{y}{x}\frac{(gx)^{\frac{1}{2}}}{Nx^k}. \tag{6}$$

Then (2) and (3) are transformed into

$$(k+1)f\frac{\partial^2 f}{\partial\eta^2} - \frac{1}{2}\left(\frac{\partial f}{\partial\eta}\right)^2 + 1 - p + (k+\tfrac{1}{2})\,\eta\frac{\partial p}{\partial\eta} + (k-\tfrac{1}{2})\,\xi\left[\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial\xi} - \frac{\partial p}{\partial\xi}\right] = 0, \tag{7}$$

$$\frac{\partial p}{\partial\eta} = -\xi\cot\alpha + \xi^2\left[(k-\tfrac{1}{2})(k+1)f\frac{\partial f}{\partial\eta} - \tfrac{1}{2}k(2k+1)\,\eta\left(\frac{\partial f}{\partial\eta}\right)^2 + \tfrac{1}{2}(k+1)(2k+1)\,\eta f\frac{\partial^2 f}{\partial\eta^2}\right] + O(\xi^3). \tag{8}$$

Here ξ clearly is a small parameter, while the variable η is of order unity in the film. The boundary conditions (4) are easily translated into

$$\partial f/\partial\eta = 0, \quad f = -1/(k+1) \quad \text{at} \quad \eta = 0, \tag{9}$$

while at the outer edge we have

$$f = 0, \quad p = 0 \quad \text{at} \quad \eta = \eta^e(\xi), \tag{10}$$

where η^e is unknown beforehand and must be determined by the analysis. For $\xi = 0$, (7) and (8) reduce to simple equations that can be solved analytically. This gives $p = 0$ and

$$\eta\sqrt{2} = \frac{1}{k+1}\int_{-(k+1)f_0}^{1} \frac{dr}{(1-r^{1/(k+1)})^{\frac{1}{2}}}. \tag{11}$$

From (10) and (11) the first approximation of the outer edge can be obtained as follows:

$$\eta_0^e = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}. \tag{12}$$

The solution is particularly simple for $k = 0$ and $k = -\frac{1}{2}$:

$$k = 0: \quad f_0 = \tfrac{1}{2}\eta^2 - 1, \quad \eta_0^e = \sqrt{2}; \tag{13}$$

$$k = -\tfrac{1}{2}: \quad f_0 = -2\cos\left(\frac{\eta}{\sqrt{2}}\right), \quad \eta_0^e = \frac{\pi}{\sqrt{2}}. \tag{14}$$

The first of these represents the important case of uniform blowing. It is seen that this solution is valid downstream since ξ tends to zero as x tends to infinity.

In order to find higher approximations one would expect that a simple regular expansion in powers of ξ should suffice. However, on doing so one would run very soon into severe difficulties, which are caused by the singular behaviour of (11) at $\eta = \eta_0^e$. Indeed, except for the simple cases (13) and (14), the zeroth-order solution is singular at $\eta = \eta_0^e$. As the second approximation includes an adverse pressure gradient represented by the second momentum equation one can expect that the actual outer edge is farther away from the wall than $\eta = \eta_0^e$. But for values of η larger than η_0^e the solution of (11) is no longer valid. A regular expansion thus fails to give the correction at the outer edge. Lighthill's technique of strained co-ordinates is especially suited to handle problems of this type. The basic feature of the method of strained co-ordinates is to introduce the strained independent variable $\bar{\eta}$ which is related to η in the following way:

$$\eta = \bar{\eta} + \xi g_1(\bar{\eta}) + O(\xi^2). \tag{15}$$

Thus the straining is weak in the sense that it disappears when the expansion variable ξ vanishes. The function $g_1(\bar{\eta})$ has to be chosen so as to produce the desired effect: to dispel the singularity from the flow region. The functions f and p are now made dependent upon $\bar{\eta}$ rather than on η and this can be stressed by writing these functions with a bar. These are now regularly expanded in the following way:

$$\bar{f}(\bar{\eta}, \xi) = \bar{f}_0(\bar{\eta}) + \xi \bar{f}_1(\bar{\eta}) + \ldots, \quad \bar{p}(\bar{\eta}, \xi) = 0 + \xi \bar{p}_1(\bar{\eta}) + \ldots, \tag{16}$$

where $\bar{f}_0(\bar{\eta})$ is exactly the same as (11). This conforms with the vanishing of straining as $\xi \to 0$.

Substituting (16) into (7) and (8) is a lengthy and tedious procedure so the algebraical details will not be given here. It results in the following equation for $\bar{f}_1$ and $\bar{p}_1$:

$$\begin{aligned}(k+1)\bar{f}_0\bar{f}_1''' - (k+\tfrac{1}{2})\bar{f}_0'\bar{f}_1' + (2k+\tfrac{1}{2})\bar{f}_0''\bar{f}_1 + (k+\tfrac{1}{2})(\bar{\eta}\bar{p}_1' - \bar{p}_1) \\ -(k+1)\bar{f}_0\bar{f}_0'g_1'' + \{(k+\tfrac{1}{2})(\bar{f}_0')^2 - 2(k+1)\bar{f}_0\bar{f}_0''\}g_1' = 0,\end{aligned} \tag{17}$$

$$\bar{p}_1' = -\cot\alpha. \tag{18}$$

The pressure $\bar{p}_1$ is easily obtained from (18) by requiring that $\bar{p}_1$ vanish at the outer edge (only $O(\xi)$ are considered). This gives

$$\bar{p}_1 = (\eta_0^e - \bar{\eta})\cot\alpha. \tag{19}$$

Equation (17) can now be written in the following way:

$$(k+1)\bar{f}_0 G'' - (k+\tfrac{1}{2})\bar{f}_0' G' + (2k+\tfrac{1}{2})\bar{f}_0'' G = (k+\tfrac{1}{2})\eta_0^e \cot\alpha, \tag{20}$$

where $G = \bar{f}_1 - \bar{f}_0' g_1$. If we observe that $G = \bar{f}_0'$ satisfies the homogeneous part of (20) it is easy to find a second solution to the homogeneous equation that is linearly independent of $G_{\mathrm{I}} = \bar{f}_0'$:

$$G_{\mathrm{II}} = \bar{f}_0'(\bar{\eta}) \int_{\bar{\eta}}^{\eta_0^e} \frac{\{-(k+1)\bar{f}_0(t)\}^{(2k+1)/(2k+2)}}{\{\bar{f}_0'(t)\}^2}\, dt. \tag{21}$$

From the zeroth order of equation (7) with $\bar{p}_0 = 0$ and its solution (11) the following identities may be obtained:

$$\{-(k+1)\bar{f}_0\}^{1/(k+1)} = 1 - \frac{(\bar{f}_0')^2}{2}; \quad \bar{f}_0'' = \left\{1 - \frac{(\bar{f}_0')^2}{2}\right\}^{-k}. \tag{22}$$

With (22), (21) can be put into the convenient form (use $\bar{f}_0'(\eta_0^e) = \sqrt{2}$)

$$G_{\mathrm{II}} = 1 + \bar{f}_0'(\bar{\eta}) \int_{\bar{f}_0'(\bar{\eta})}^{\sqrt{2}} \frac{(1-\frac{1}{2}r^2)^{2k+\frac{1}{2}} - 1}{r^2}\, dr - \frac{\bar{f}_0'(\bar{\eta})}{\sqrt{2}} = G_{\mathrm{III}} - \frac{\bar{f}_0'(\bar{\eta})}{\sqrt{2}}. \tag{23}$$

Since it is also possible to find an analytical solution of the inhomogeneous equation, the general solution of (20) can be given as

$$\bar{f}_1 = G + \bar{f}_0' g_1' = (A + g_1)\bar{f}_0' + BG_{\mathrm{III}} + \{(k+\tfrac{1}{2})\,\bar{\eta}\bar{f}_0' - (k+1)\bar{f}_0\}\,\eta_0^e \cot\alpha, \tag{24}$$

where A and B are constants of integration. These constants have to be determined by applying the boundary conditions at $\eta = 0$ (9). The functions $\bar{f}_0, \bar{f}_1$, etc., however, are dependent on $\bar{\eta}$ rather than on η. By inverting (15) the value of $\bar{\eta}$ at $\eta = 0$ is easily obtained: $\bar{\eta}(\eta = 0) = -g_1(0)\,\xi +$ higher orders. Upon evaluating $\bar{f}_0\{-g_1(0)\,\xi\} + \xi\bar{f}_1\{-g_1(0)\,\xi\}$ up to $O(\xi)$, using (11) and (24) we have

$$-\frac{1}{k+1} = \left[-\frac{1}{k+1} + O(\xi^2)\right] + \xi[B + \eta_0^e \cot\alpha + O(\xi)], \tag{25}$$

so that $B = -\eta_0^e \cot\alpha$. Applying the second condition $\partial f/\partial\eta = 0$ at $\eta = 0$ is somewhat more involved. We have to use

$$\frac{\partial f}{\partial \eta} = \frac{\partial \bar{f}}{\partial \bar{\eta}}(1 - \xi g_1' + \ldots) = \bar{f}_0' + \xi(\bar{f}_1' - g_1'\bar{f}_0') + \ldots. \tag{26}$$

Evaluating the right-hand side of (26) for $\bar{\eta} = -g_1(0)\,\xi$, expanding for small values of ξ, and requiring the coefficient of ξ to vanish, yield

$$A = -B\int_0^{\sqrt{2}} \frac{(1-\frac{1}{2}r^2)^{2k+\frac{1}{2}} - 1}{r^2}\, dr = -\frac{1}{\sqrt{2}}\left[\pi^{\frac{1}{2}} \frac{\Gamma(2k+\frac{3}{2})}{\Gamma(2k+1)} - 1\right]\eta_0^e \cot\alpha. \tag{27}$$

Note that the determination of the constants A and B is independent of the straining function g_1.

The location of the outer edge can be obtained by substituting

$$\bar{\eta}^e = \bar{\eta}_0^e + \bar{\eta}_1^e\,\xi + \ldots \quad (\bar{\eta}_0^e = \eta_0^e)$$

into $\bar{f}_0 + \xi\bar{f}_1$ and requiring the result to equal zero. This gives

$$\bar{\eta}_1^e = \left\{\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(2k+\frac{3}{2})}{\Gamma(2k+1)}\eta_0^e - (k+\tfrac{1}{2})\,(\eta_0^e)^2\right\}\cot\alpha - g_1(\bar{\eta}_0^e). \tag{28}$$

Upon substituting the two-term expansion of $\bar{\eta}^e$ into (15), the true position of the outer edge is found up to $O(\xi)$:

$$\eta^e = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}\left[1 + \xi\left\{\frac{\Gamma(2k+\frac{3}{2})}{\Gamma(2k+1)} - (k+\tfrac{1}{2})\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}\right\}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\cot\alpha\right], \tag{29}$$

which is again independent of g_1, as it should be. It is rather easy to prove that the coefficient of ξ is positive for arbitrary values of k. Thus, as anticipated, the true position of the outer edge is farther away from the wall than the zeroth order. Therefore, if no straining is applied, i.e. if $\bar{\eta}$ is identified with η, the solution will not be defined at the outer edge. The straining should be such that $\bar{\eta}^e$ is not *inside* the flow field, i.e. $\bar{\eta}_1^e$ should not be positive; otherwise the straining is quite arbitrary. It seems convenient to choose

$$g_1(\bar{\eta}) = (2k+\tfrac{1}{2})\,\eta_0^e \cot\alpha \int_0^{\bar{f}_0'(\bar{\eta})} (1-\tfrac{1}{2}r^2)^{2k-\frac{1}{2}}\,dr \quad (k > -\tfrac{1}{4}). \tag{30}$$

From (24) and (28) we then obtain

$$\left.\begin{aligned} \bar{f}_1 &= [(k+\tfrac{1}{2})\,\bar{\eta}\bar{f}_0' - (k+1)\bar{f}_0 - \{1-\tfrac{1}{2}(\bar{f}_0')^2\}^{2k+\frac{1}{2}}]\,\eta_0^e\cot\alpha, \\ \bar{\eta}_1^e &= -(k+\tfrac{1}{2})\,(\eta_0^e)^2\cot\alpha. \end{aligned}\right\} \tag{31}$$

It may be in order to give the results for the important case of uniform blowing. For $k = 0$ the following results may be obtained:

$$\bar{f}(\bar{\eta},\xi) = (\tfrac{1}{2}\bar{\eta}^2-1)+\xi\{1-(1-\tfrac{1}{2}\bar{\eta}^2)^{\frac{1}{2}}\}\sqrt{2}\cot\alpha+\ldots, \tag{32a}$$

$$\bar{p}(\bar{\eta},\xi) = 0+\xi(\sqrt{2}-\bar{\eta})\cot\alpha+\ldots, \tag{32b}$$

$$\eta^e = \sqrt{2}+\xi(\tfrac{1}{2}\pi-1)\cot\alpha+\ldots, \tag{32c}$$

$$\eta = \bar{\eta}+\xi\arcsin(\bar{\eta}/\sqrt{2})\cot\alpha+\ldots. \tag{32d}$$

3. Almost horizontal wall

In the examples that have been discussed up to now, the pressure equals zero in the fundamental term. This is due to the fact that the second momentum equation only adds effects of higher order. If the plate is almost horizontal, i.e. if $\alpha \sim 0$ or $\cot\alpha \gg 1$, this can no longer be true. The pressure effect of the second momentum equation is so important, that it has to be included in the main flow. This flow is now described by (1) and (2) and by

$$0 = -\frac{\partial}{\partial y}\left(\frac{\tilde{p}}{\rho}\right) - g\cot\alpha. \tag{33}$$

Upon integration this gives for the pressure

$$\frac{\tilde{p}}{\rho} = \frac{\tilde{p}^e}{\rho} + (y^e-y)\,g\cot\alpha. \tag{34}$$

Substitution of (34) into the first inviscid momentum equation leads to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\left(1-\frac{dy^e}{dx}\cot\alpha\right). \tag{35}$$

If $dy^e/dx > 0$, i.e. if the film becomes thicker downstream, which usually is the case, the forcing term on the right-hand side of (35) becomes smaller. We can imagine that for values of α that are sufficiently near zero, the force may vanish or even become negative. This will halt or reverse the flow. Within the context of this paper it does not seem to be realistic to consider these reverse flows. An example to be given presently will show that a certain value of α exists, below which no solution to the problem can be found. This leads us to the conclusion that backward flows are not described by the simple equation (35).

(a) *A similarity solution*

Equations (1) and (35) admit a simple similarity solution if the injection rate is proportional to $x^{\frac{1}{2}}$. In this case we use the following substitutions:

$$\psi = \tfrac{2}{3}Nx^{\frac{3}{2}}f(\eta); \quad \eta = \frac{y\,g^{\frac{1}{2}}}{x\,N}, \quad y^e = \gamma x \tag{36}$$

to obtain an ordinary differential equation to describe the flow

$$2(f')^2 - 6ff'' = 9(1-\gamma\cot\alpha), \tag{37}$$

where f has to satisfy the following boundary conditions:

$$f(0) = -1, \quad f'(0) = 0, \quad f(\gamma g^{\frac{1}{2}}/N) = 0. \tag{38}$$

Equation (37) may be integrated once, yielding

$$|\tfrac{9}{2}(1-\gamma\cot\alpha) - (f')^2| = \tfrac{9}{2}\,|1-\gamma\cot\alpha|\;|f|^{\frac{2}{3}}, \tag{39}$$

which satisfies the boundary conditions at $\eta = 0$. It now follows that

$$\tfrac{9}{2}(1-\gamma\cot\alpha) \geqslant 0.$$

If this condition is not fulfilled the function f can never attain the value zero, which it must do at $y = y^e$. Since f should not be equal to zero at a location other than the outer edge, we conclude that always $\frac{9}{2}(1-\gamma\cot\alpha)-(f')^2 \geqslant 0$. We can now integrate (39) further and obtain

$$\begin{aligned} 3\eta\,(\tfrac{1}{2}(1-\gamma\cot\alpha))^{\frac{1}{2}} &= \int_{-f}^{1} \frac{dt}{(1-t^{\frac{2}{3}})^{\frac{1}{2}}} \\ &= \tfrac{3}{2}[\tfrac{1}{2}\pi - \arctan\{(-f)^{\frac{1}{3}}/[1-(-f)^{\frac{2}{3}}]\}^{\frac{1}{2}} + \{(-f)^{\frac{1}{3}}[1-(-f)^{\frac{2}{3}}]\}^{\frac{1}{2}}]. \end{aligned} \tag{40}$$

The unknown parameter γ is obtained by application of the third condition of (38). With (40) this gives

$$\gamma(1-\gamma\cot\alpha)^{\frac{1}{2}} = \tfrac{1}{4}\pi N\{2/g\}^{\frac{1}{2}}. \tag{41}$$

In order to study the dependence of γ on α it is necessary to note that g, being the longitudinal component of the force of gravity $\tilde{g}$, also depends on α: $g = \tilde{g}\sin\alpha$. Rewrite (41) as

$$\phi(r) = r^3 - \frac{\sin\alpha}{m}r + \frac{\cos\alpha}{m} = 0, \quad m = \frac{\pi^2N^2}{8\tilde{g}}, \quad r = \frac{1}{\gamma}, \tag{42}$$

which is the standard form for cubic equations. It is known (Upspensky 1948) that the solutions to (42) depend strongly on the value of the discriminant

$$\Delta = \frac{\cos^2\alpha}{4m^2} - \frac{\sin^3\alpha}{27m^3}. \tag{43}$$

If $\Delta > 0$ there is only one real solution. One can easily convince oneself that this solution is negative ($\phi(0) > 0$ but $\phi \to -\infty$ as $r \to -\infty$). The original equation (41) does not allow negative values of γ, which leads to the conclusion that for those values of α for which the discriminant is positive the problem does not have a solution. It is easily seen that this situation is attained for certain values of α that are small enough (small slope of the plate) or for large enough values of m (large injection rates or low gravity fields).

If $\Delta < 0$ there are three real solutions to (42), one of which is again negative. In order to point out which of these positive values is the correct one, let us return to (41). One can readily convince oneself, by employing a graphical method, that (41) yields two positive values of γ. The smaller of these values decreases when the injection rate (N) decreases or when the force of gravity increases. This is a physically realistic behaviour since one expects thinner films under those conditions. The other root displays an opposite behaviour so that we have to choose the smaller of the two roots as our solution. By using classical methods, this solution can be obtained:

$$\gamma = \left\{\frac{3m}{4\sin\alpha}\right\}^{\frac{1}{2}} \Big/ \cos\tfrac{1}{3}\left[\pi - \arctan\left\{\frac{4\sin^3\alpha}{27m\cos^2\alpha} - 1\right\}^{\frac{1}{2}}\right]. \tag{44}$$

The lower bound on α and the upper bound on m that exist for the validity of (44) are given by $\Delta = 0$. From (43) we find that these limiting conditions should be obtained from

$$s^3 - s - \frac{4}{27m} = 0, \quad s = \frac{1}{\sin\alpha}. \tag{45}$$

The discriminant of this equation

$$\left(\frac{2}{27m}\right)^2 - \frac{1}{27} \tag{46}$$

is seen to be positive if $m < 2\sqrt{3}/9$. In that case there is only one real root of (45) and this can be found by using Cardan's formula

$$\sin\alpha_m = \frac{1}{s_m} = \left[\left\{\frac{2}{27m}\left(1 + \left(1 - \frac{27m^2}{4}\right)^{\frac{1}{2}}\right)\right\}^{\frac{1}{3}} + \left\{\frac{2}{27m}\left(1 - \left(1 - \frac{27m^2}{4}\right)^{\frac{1}{2}}\right)\right\}^{\frac{1}{3}}\right]^{-1},$$
$$\text{for} \quad m < \frac{2\sqrt{3}}{9}. \tag{47}$$

If, on the other hand, the discriminant (46) is positive, there will be three real roots. It is easily proved that two of these are negative. The only positive root yields

$$\sin\alpha_m = \frac{1}{s_m} = \frac{\sqrt{3}}{2\cos\left\{\frac{1}{3}\arctan\left(\frac{27m^2}{4} - 1\right)^{\frac{1}{2}}\right\}}, \quad \text{for} \quad m > \frac{2\sqrt{3}}{9}. \tag{48}$$

From both (47) and (48) we find for the transition case

$$\sin\alpha_m = \sqrt{3}/2 \quad \text{or} \quad \alpha_m = 60°, \quad \text{for} \quad m = 2\sqrt{3}/9. \tag{49}$$

Thus, the value $m = 2\sqrt{3}/9$ already represents a case of very strong blowing that probably is not realistic. Only the very small values of m seem to be realizable. By evaluating (47) for very small values of m, we find an approximate formula for the limiting surface inclination:

$$\alpha_m \sim \tfrac{3}{2}(2m)^{\frac{1}{3}}. \tag{50}$$

Finally, for a given inclination of the porous surface, the maximum value of m is given directly by (45).

(*b*) *Arbitrary injection rates*

It is possible to integrate (35) for arbitrary injection rates. To that end we use von Mises's transformation of co-ordinates in which the velocity is taken as a function of x and the stream function ψ. If we apply this technique to (35) we can integrate once with respect to x yielding

$$u^2(x,\psi) = 2g\{x - i\psi_w(\psi) + [y^e\{i\psi_w(\psi)\} - y^e(x)]\cot\alpha\}. \tag{51}$$

Here $i\psi_w$ is the inverse function of ψ_w, which is the value of the stream function at the wall. The solution of (51) satisfies the boundary conditions at the wall. Equation (51) can be readily put into the form

$$y(2g)^{\frac{1}{2}} = \int_{\psi_w(x)}^{\psi} \frac{d\psi}{\{x - i\psi_w(\psi) + [y^e\{i\psi_w(\psi)\} - y^e(x)]\cot\alpha\}^{\frac{1}{2}}}.$$

By choosing $i\psi_w(\psi)$ for a new variable, this yields

$$y = \frac{1}{(2g)^{\frac{1}{2}}}\int_{i\psi_w(\psi)}^{x} \frac{v_w(t)\,dt}{[x - t + \{y^e(t) - y^e(x)\}\cot\alpha]^{\frac{1}{2}}}, \tag{52}$$

where we have introduced the injection velocity $v_w(x) = -\psi_w'(x)$.

Equation (52) still cannot be considered to be the solution of the problem, as it contains the unknown outer edge $y^e(x)$. However, (52) yields a singular non-linear integral equation for $y^e(x)$, if it is evaluated at $\psi = 0$:

$$y^e(x) = \frac{1}{(2g)^{\frac{1}{2}}}\int_0^x \frac{v_w(t)\,dt}{[x - t + \{y^e(t) - y^e(x)\}\cot\alpha]^{\frac{1}{2}}}. \tag{53}$$

Here we have used $i\psi_w(0) = 0$ since $\psi_w(0) = 0$. If equation (53) has been solved, the result can be substituted into (52) yielding the solution for arbitrary blowing.

The validity of (53) can be checked easily for $k = \frac{1}{2}$ by substitution and comparing with (41).

As a further example, let us consider the case of uniform blowing: $v_w(t)$ is a constant. In this case we use a device of Cole & Aroesty (1968) which transforms (53) into an integral equation of the Abel type. Put

$$x - y^e(x)\cot\alpha = \Omega; \quad t - y^e(t)\cot\alpha = \omega. \tag{54}$$

From (53) we then obtain

$$y^e\{x(\Omega)\} = \frac{v_w}{(2g)^{\frac{1}{2}}}\int_0^{\Omega} \frac{(dt/d\omega)\,d\omega}{(\Omega - \omega)^{\frac{1}{2}}}. \tag{55}$$

This is indeed Abel's integral equation. We now proceed by inverting this equation using a standard technique (Pogorzelski 1966) which gives

$$\frac{v_w}{(2g)^{\frac{1}{2}}}\frac{dt}{d\omega} = \frac{1}{\pi}\int_0^{\omega} \frac{(dy^e/dx)\,(dx/d\Omega)\,d\Omega}{(\omega - \Omega)^{\frac{1}{2}}}. \tag{56}$$

Using the original variables, but interchanging their roles, we obtain the following:

$$\frac{v_w \pi}{(2g)^{\frac{1}{2}}} = \left(1 - \frac{dy^e}{dx}\cot\alpha\right)\int_0^x \frac{(dy^e/dt)\,dt}{[x - t + \{y^e(t) - y^e(x)\}\cot\alpha]^{\frac{1}{2}}}. \tag{57}$$

By writing the numerator of the integrand as

$$\left(\frac{dy^e}{dt} - \frac{1}{\cot\alpha}\right) + \frac{1}{\cot\alpha}$$

we can split up the integral of (57) into two separate integrals. The first of these can be integrated directly, while the second can be expressed in y^e by using (53) for constant v_w. The result is the following non-linear first-order differential equation for $y^e(x)$:

$$\left.\begin{aligned} \pi\epsilon &= (1-\delta')\{(\delta/\epsilon) - 2(\chi-\delta)^{\frac{1}{2}}\}, \\ \epsilon &= v_w \cot\alpha/(2gl)^{\frac{1}{2}}, \quad \delta(\chi) = y^e \cot\alpha/l, \quad \chi = x/l, \end{aligned}\right\} \tag{58}$$

where l is an auxiliary reference length. By writing $\delta(\chi) = \chi - z(\chi)$ this equation may become a linear equation if the roles of z and χ are interchanged:

$$\frac{d\chi}{dz} - \frac{\chi}{\pi\epsilon^2} = -\frac{z}{\pi\epsilon^2} - \frac{2}{\pi\epsilon} z^{\frac{1}{2}}. \tag{59}$$

The general solution is

$$\chi = A\, e^{z/\pi\epsilon^2} + z + 2\epsilon z^{\frac{1}{2}} + \pi\epsilon^2 - \pi\epsilon^2 e^{z/\pi\epsilon^2} \operatorname{erf}(z/\pi\epsilon^2)^{\frac{1}{2}}. \tag{60}$$

When choosing the value of the constant of integration, we have to take into account that the inviscid film resulting from uniform blowing is not valid at the leading edge, but its validity starts sufficiently far downstream. Thus, the constant A has to be chosen in such a way so as to satisfy certain downstream conditions. It is seen that, by choosing $A = \pi\epsilon^2$, the behaviour for $\chi \to \infty$ differs considerably from that obtained by taking any other value of A. We will proceed by taking $A = \pi\epsilon^2$ and we will prove this to be the correct choice by comparing the result with the findings of §2. Equation (60) can now be written

$$\chi = z + 2\epsilon z^{\frac{1}{2}} + \pi\epsilon^2 + \pi\epsilon^2 e^{z/\pi\epsilon^2} \operatorname{erfc}(z/\pi\epsilon^2)^{\frac{1}{2}}, \tag{61}$$

where erfc denotes the complementary error function. Evidently for $\chi \to \infty$ the variable z will tend to infinity. By using an asymptotic expansion for the complementary error function, (61) can be inverted for large values of χ, which gives

$$z = \chi - 2\epsilon\chi^{\frac{1}{2}} + (2-\pi)\epsilon^2 + O(\chi^{-\frac{1}{2}}). \tag{62}$$

Using again the original variables and (6) this may be transformed into

$$\eta^e = \sqrt{2} + (\tfrac{1}{2}\pi - 1)\,\xi\cot\alpha + O(\xi^2), \tag{63}$$

which agrees with (32*c*).

4. Concluding remarks

In the present work, several aspects of inviscid film flow with fluid injection have been considered, and a number of analytical solutions have been presented. Due to the inviscid nature of the flow, the normal derivative of the tangential velocity does not vanish at the outer edge of the film. If viscous effects are taken into account, this normal derivative should vanish in order to ensure vanishing shear stress at the outer edge. Including this effect requires application of a singular perturbation technique of the inner and outer expansion type. Here the inviscid layer would be the outer layer, as it comprises the major part of the flow field. In the very thin 'inner' layer, which is located at the outer edge of the inviscid layer, the velocity gradient changes rapidly from its non-zero inviscid value to zero at the true outer edge of the film. Work on this phenomenon is presently in progress and the results are expected to be published in a subsequent paper.

The author is indebted to the National Research Council of Canada for financial support.

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